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On Weakly Conjugate ζ-Permutable Subgroups of Finite Groups

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Abstract

Let C $\neq\phi$ be subset of $\mathcal{G}$ and ζ represents complete set of Sylow subgroups of $\mathcal{G}$. Let Ƕ $\leq \mathcal{G}$ is C- ζ permutable (conjugate ζ permutable) subgroup within $\mathcal{G}$ if for Ƕ$^X\mathcal{G}_q$= $\mathcal{G}_q$Ƕ^X $\exists$ x∈C, $\forall$ $\mathcal{G}_q \in \zeta$ be a finite group. A subgroup Ƕ$\leq\mathcal{G}$ is known as weakly Conjugate ζ-permutable within $\mathcal{G}$ when $\exists$, $\mathcal{K}\leq \mathcal{G}$ s.t $\mathcal{G}$= Ƕ$\mathcal{K}$ and Ƕ$\cap\mathcal{K}\leq$Ƕ$_{c\zeta}$ where <Ƕ$_{c\zeta}$>$\subseteq$Ƕ are conjugate ζ permutable within $\mathcal{G}$. Our main goal: $\mathcal{G}$ is supersolvable when maximal subgroups of $\mathcal{G}_q\cap$Ƒ$^*(\mathcal{G})$ are weakly C-ζ-permutable within $\mathcal{G}$, for each $\mathcal{G}_q \in \zeta$, where Ƒ$^*(\mathcal{G})$ denote generalized Fitting subgroup of $\mathcal{G}$. Moreover, we show when, $\mathcal{G}$ ∈ Ƒ, Ƒ is saturated formation consist of every supersolvable groups if and only if Ƕ⊴$\mathcal{G}$ s.t $\mathcal{G}$/Ƕ∈ Ƒ and maximal subgroups of $\mathcal{G}_q\cap$Ƒ*(Ƕ) becomes weakly C- ζ-permutable Within $\mathcal{G}$, for each $\mathcal{G}_q \in \zeta$.

Keywords: Weakly C- ζ-permutable, supersolvable groups, maximal subgroups, finite groups

Introduction

All groups assumed to be finite. Majority of notation is level and locate in Ballester Bolinches et al. [2], Doerk and Hawkes [3]. Moreover, $\mathcal{G}$ represents set of different primes dividing $|\mathcal{G}|$and $\mathcal{G}_q$ denotes Sylow q-subgroup of group $\mathcal{G}$ for few prime $q \in \pi(\mathcal{G})$. Remember: a subgroup $H_1 \leq \mathcal{G}$ is known as S-permutable whenever it permutes along each Sylow subgroup of $\mathcal{G}$. This idea was initiate by Kegel [7], who said these subgroups *S*-quasi normal and different authors' works on this. Asaad, Heliel [1] enhanced *S*-permutability by changing permutability along ζ. When every q divides $|\mathcal{G}|$, ζ has only one $\mathcal{G}_q$. A subgroup $H_1 \leq \mathcal{G}$ becomes ζ-permutable within $\mathcal{G}$ when H_1 permutes along each member of ζ. [12] $H_1 \leq \mathcal{G}$ is known as C-ζ permutable (conjugate ζ permutable) when some x ∈ C s.t ${H_1}^X\mathcal{G}_q$= ${\mathcal{G}_q}^X$, for each $\mathcal{G}_q \in \zeta$. By Wang [11], a subgroup $H_1 \leq \mathcal{G}$ becomes c-normal within $\mathcal{G}$ whenever $H_2 \trianglelefteq \mathcal{G}$ s.t H_1H_2=$\mathcal{G}$ and $H_1\cap H_2 \leq (H_1)_{\mathcal{G}}$, where $(H_1)_{\mathcal{G}}$=$core_{\mathcal{G}}(H_1)$. There exist groups with ζ-permutable which are not c-normal and vice versa hold; sight Examples 1, 2 & 3 of [4]. Furthermore, ζ-permutability & c-normality both extended by [4] as: a subgroup Ƕ$\leq\mathcal{G}$ becomes weakly ζ-permutable within $\mathcal{G}$ whenever $\mathcal{G}$=Ƕ$\mathcal{K}$ for $\mathcal{K} \lhd \mathcal{G}$ (subnormal) and Ƕ$\cap\mathcal{K}$ $\leq$Ƕ$_\zeta$, where ⟨Ƕ$_\zeta$⟩⊆Ƕ are ζ–permutable within $\mathcal{G}$. By embedding characterization of this new subgroup, authors in [4] observed that when every maximal subgroups of certain or each belong to ζ for few normal subgroup are weakly ζ-permutable within $\mathcal{G}$. Where they find some results and extend many present results within literature. This paper perhaps observed as enhancement of [4]; in short, coming results exist in [4]:

Definition: A subgroup Ƕ$\leq\mathcal{G}$ be weakly Conjugate ζ-permutable whenever $\mathcal{K}\leq\mathcal{G}$ s.t Ƕ$\mathcal{K}$=$\mathcal{G}$ and Ƕ$\cap\mathcal{K}\leq$Ƕ$_{c\zeta}$, where ⟨Ƕ$_{c\zeta}$⟩⊆ Ƕ are conjugate ζ-permutable within $\mathcal{G}$.

Example: Consider $\mathcal{G}=A_4$, the alternating group of degree 4, Ƕ=⟨(134)⟩, C=⟨(142)⟩, ζ=V$_4$, ⟨(123)⟩, where $\mathcal{K}$=V$_4$ *is the Klein* 4-*group.* Clearly $\mathcal{G}$=Ƕ$\mathcal{K}$, Ƕ∩$\mathcal{K}$≤Ƕ$_{c\zeta}$. Therefore Ƕ becomes C-ζ-permutable and hence weakly conjugate ζ-permutable in $\mathcal{G}$.

Theorem 1.1. *Suppose* $|\mathcal{G}|$ is divisible by prime q. *When maximal subgroups of* $\mathcal{G}_q \in \zeta$ *are weakly Conjugate* ζ*-permutable within* $\mathcal{G}$*, so* $\mathcal{G}$ *becomes q-nilpotent.*

Theorem 1.2. *Suppose* $\mathcal{F}$ *carrying all types of supersolvable groups* µ. So, *coming statements are identical:*

1. $\mathcal{G} \in$ Ƒ.
2. *For a solvable* $\mathcal{K} \trianglelefteq \mathcal{G}$ *s.t* $\mathcal{G}/\mathcal{K} \in$ Ƒ *and maximal subgroups of Syl(*Ƒ(Ƕ)),*becomes weakly Conjugate* ζ*-permutable within* $\mathcal{G}$.

The major purpose to proceeds the beyond results next for proving:

Theorem 1.3. *When maximal subgroups of* $\mathcal{G}_q \cap$ Ƒ$^*(\mathcal{G})$ *form weakly Conjugate* ζ*-permutable ,* $\mathcal{G}_q \in \zeta$ *so* $\mathcal{G}$ *becomes supersolvable.*

Theorem 1.4. For Ƒ & ζ *these two statements are identical:*

1. $\mathcal{G} \in$ Ƒ.
2. *For* $\mathcal{K} \trianglelefteq \mathcal{G}$ *s.t* $\mathcal{G}/\mathcal{K} \in$ Ƒ *and maximal subgroups of* $\mathcal{G}_q \cap$ Ƒ$^*(\mathcal{K})$ *form weakly Conjugate* ζ*-permutable within* $\mathcal{G}$*,* $\forall\, \mathcal{G}_q \in \zeta$.

Theorems 1.3, 1.4 modified and enhanced few familiar results of literature (look at Corollaries 3.1, 3.2, 3.3, 3.4 & 3.5). Remember, Ƒ$^*(\mathcal{G})$ be set of each y∈ $\mathcal{G}$ which generate an inner automorphism on each chief factor of $\mathcal{G}$. Ƒ$^*(\mathcal{G})$ is an main characteristic subgroup of $\mathcal{G}$ and generalization of Ƒ$(\mathcal{G})$ (Fitting subgroup). From [5, X 13], Ƒ$^*(\mathcal{G})\neq 1$ if $\mathcal{G} \neq 1$. For the characterization of Ƒ$^*(\mathcal{G})$ and saturated formation see [5, X 13]& [3].

2. Some important lemmas

Lemma 2.1. Suppose a soluble Ƕ⊴Ƒ$^*(\mathcal{G})$. When maximal subgroups of $\mathcal{G}_q \cap$ Ƒ$^*(\mathcal{G})$ are conjugate ζ-permutable in $\mathcal{G}$, $\forall\, \mathcal{G}_q \in \zeta$ so $\mathcal{G}$ becomes supersolvable.

Proof. Follow [12, Corollary 3.11]. ▭

Lemma 2.2. Consider a group $\mathcal{G}$. Next:

1. When $\mathcal{K} \trianglelefteq \mathcal{G}$ indicate Ƒ$^*(\mathcal{K})\leq$Ƒ$^*(\mathcal{G})$.
2. Ƒ*(Ƒ$^*(\mathcal{G})$)=Ƒ$^*(\mathcal{G})\geq$Ƒ$(\mathcal{G})$; when Ƒ$^*(\mathcal{G})$ be soluble, Ƒ$^*(\mathcal{G})$=Ƒ$(\mathcal{G})$.
3. Assume $\mathcal{T} \trianglelefteq \mathcal{G} \subseteq$ Z$(\mathcal{G})$, subsequently Ƒ$^*(\mathcal{G}/\mathcal{T})$= Ƒ$^*(\mathcal{G})/\mathcal{T}$.
4. Let $\mathcal{K} \trianglelefteq \mathcal{G} \subseteq \Phi(\mathcal{G})$, laterally F$^*(\mathcal{G}/\mathcal{K})$= Ƒ$^*(\mathcal{G})/\mathcal{K}$.

Proof. See [12, Lemma 2.7]. ▭

Lemma 2.3. Suppose $H_1, H_2 \leq \mathcal{G}$ s.t $H_1 \trianglelefteq \mathcal{G}$. Then

1. When $H_1 \leq H_2$ & H_1 be weakly Conjugate ζ-permutable within $\mathcal{G}$, then H_1 is weakly conjugate

$\zeta \cap H_2-$ permutable within H_2 .

2. When $H_2 \leq H_1$, next H_1 is weakly Conjugate ζ-permutable $\mathcal{G} \Leftrightarrow H_1/H_2$ is weakly conjugate $\zeta H_2/H_2-$ permutable within $\mathcal{G}/H_2$.
3. When $(|H_1|, |H_2|)=1$ and H_1 is weakly Conjugate ζ-permutable within $\mathcal{G}$, laterally H_1H_2/H_2 is weakly conjugate $\zeta H_2/H_2-$ permutable within $\mathcal{G}/H_2$.
4. *When H_1 is a q-subgroup of $\mathcal{G}$ for few prime q s.t H_1 is weakly Conjugate ζ-permutable within $\mathcal{G}$ however H_1 not ζ-permutable within $\mathcal{G}$, then one can find $H_2 \trianglelefteq \mathcal{G}$ with $|\mathcal{G}: H_2|= q$ & $\mathcal{G}= H_1H_2$.*

Proof. By use similar arguments as in [4, Lemma 2.3].

Lemma 2.4. For *ζ and Ƒ carrying* μ. T*he coming statements are identical:*

1. $\mathcal{G} \in$ Ƒ.
2. *Whenever a solvable $H_1 \trianglelefteq \mathcal{G}$ s.t $\mathcal{G}/H_1 \in$ Ƒ and the maximal Syl(Ƒ(H_1)) are weakly Conjugate ζ-permutable within $\mathcal{G}$.*

Proof. Same as in [4, Theorem1.6]. □

Lemma 2.5. *Suppose $\mathcal{K} \trianglelefteq \mathcal{G}$ and Q be q-subgroup of $\mathcal{G}$ for few prime q. So.*

1. *When $\mathcal{T}/\mathcal{K}$ is a maximal subgroup of $Q\mathcal{K}/$, then $\mathcal{T} = (\mathcal{T} \cap Q)\, \mathcal{K}$, Where $\mathcal{T} \cap Q \leq Q$ (maximal).*
2. *When* $(|Q|, |\mathcal{K}|)=1$ *and maximal subgroups of Q becomes weakly conjugate ζ- permutable within $\mathcal{G}$, next maximal subgroups of $Q\mathcal{K}/\mathcal{K}$ are weakly conjugate $\zeta\, \mathcal{K}/\mathcal{K}$ -permutable within $\mathcal{G}/\mathcal{K}$.*

Proof. Look at [4, Lemma 2.4].

Lemma 2.6. *Consider $\mathcal{K} \trianglelefteq \mathcal{G}$ and nilpotent subgroup of $\mathcal{G}$ and $\mathcal{J}$ be normal q-subgroup of $\mathcal{G}$, for few prime q, with $\mathcal{J} \leq \mathcal{K}$.* When *maximal of Syl($\mathcal{K}$) are weakly Conjugate ζ-permutable within $\mathcal{G}$, so maximal of Syl($\mathcal{K}/\mathcal{J}$) are weakly conjugate $\zeta\, \mathcal{J}/\mathcal{J}$ -permutable within $\mathcal{G}/\mathcal{J}$. Moreover, when maximal subgroups of $\zeta \cap \mathcal{J}$ are weakly Conjugate ζ-permutable within $\mathcal{G}$, then maximal subgroups of $(\zeta\, \mathcal{J}/\mathcal{J}) \cap (\mathcal{K}/\mathcal{J})$ are weakly conjugate $\zeta\, Q/\mathcal{J}$ - permutable within $\mathcal{G}/\mathcal{J}$.*

Proof. Assume $\mathcal{W}$ be Sylow p-subgroup of $\mathcal{K}$. Science $\mathcal{W}$ is characteristic in $\mathcal{K}$, $\mathcal{K} \trianglelefteq \mathcal{G}$ so, $\mathcal{W} \trianglelefteq \mathcal{G}$. Suppose $q \neq p$. Since $(|\mathcal{W}|, |\mathcal{J}|) = 1$, maximal subgroups of $\mathcal{W}$ are weakly Conjugate ζ-permutable within $\mathcal{G}$, thus maximal subgroups of $Q\mathcal{W}/\mathcal{W}$ are weakly conjugate $\zeta\, \mathcal{J}/\mathcal{J}$ *-permutable* within $\mathcal{G}/\mathcal{J}$ (from Lemma 2.5(b)). So, we consider that $q=p$ and so $\mathcal{J} \leq \mathcal{W}$. Let $M/\mathcal{J} \leq \mathcal{W}/Q(maximal)$. $M \leq \mathcal{W}\ (maximal)\ from$ Lemma 2.5(a). So, $M/\mathcal{J}$ is weakly conjugate $\zeta\, \mathcal{J}/\mathcal{J}-$*permutable* within $\mathcal{G}/\mathcal{J}$ by supposition and from Lemma 2.3(b). Thus, maximal subgroups of $\mathcal{W}/\mathcal{J}$ are weakly conjugate $\zeta\mathcal{J}/\mathcal{J}$ $-$*permutable* within $\mathcal{G}/\mathcal{J}$. Thus, maximal subgroups of Syl$(\mathcal{K}/\mathcal{J})$ form weakly conjugate $\zeta\, \mathcal{J}/\mathcal{J}$ -*permutable* within $\mathcal{G}/\mathcal{J}$. Surly, $\zeta \cap \mathcal{K}$ be the set of Syl($\mathcal{K}$) science $\mathcal{K}$ is nilpotent. As $\zeta\mathcal{J}/\mathcal{J}$ is a complete set of Syl$(\mathcal{G}/\mathcal{J})$, $\mathcal{K}/\mathcal{J} \trianglelefteq \mathcal{G}/\mathcal{J}$ *and* nilpotent this implies $(\zeta\mathcal{J}/\mathcal{J}) \cap (\mathcal{K}/\mathcal{J})$ be set of Syl$(\mathcal{K}/\mathcal{J})$. Consequently, maximal subgroups of $(\zeta\, \mathcal{J}/\mathcal{J}) \cap (\mathcal{K}/\mathcal{J})$ are weakly conjugate $\zeta\, \mathcal{J}/\mathcal{J}-$*permutable* within $\mathcal{G}/\mathcal{J}$.

Lemma 2.7. *Suppose $\mathcal{K} \leq \mathcal{G}$. Then $\mathcal{K}\zeta$ is ζ-permutable within $\mathcal{G}$ and $\mathcal{K}_{\mathcal{G}} \leq \mathcal{K}_{\zeta}$*

Proof. Look [4, Lemma 2.2(a)]. □

Lemma 2.8. *Suppose $H_1, H_2 \leq \mathcal{G}$ s.t $H_2 \trianglelefteq \mathcal{G}$. If H_1 is ζ-permutable within $\mathcal{G}$ implies $H_1 \cap H_2$ is ζ-permutable* *within $\mathcal{G}$.*

Proof. Follow [4, Lemma 2.1(e)].

Lemma 2.9. *Suppose $\mathcal{K} \trianglelefteq \mathcal{G}$ ($\mathcal{K} \neq 1$) and $\mathcal{K} \cap \emptyset(\mathcal{G}) = 1$. So $F(\mathcal{K})$ of $\mathcal{K}$ be direct product of minimal normal subgroups of $\mathcal{G}$ that lies in $F(\mathcal{K})$*

Proof. Follow [9, Lemma 2.6].

3. Theorems Proof

Proof of Theorem 1.3: Consider our statement is wrong and suppose a counter example of $\mathcal{G}$ has minimal order. So next information's about $\mathcal{G}$ are correct:

(1) $F^*(\mathcal{G}) \neq \mathcal{G}$. Consider $F^*(\mathcal{G}) = \mathcal{G}$. As maximal subgroups of $\mathcal{G}_q$ become weakly conjugate ζ-permutable within $\mathcal{G}$, for all $\mathcal{G}_q \in \zeta$, Lemma 2.1 implies becomes supersolvable, a negation so $F^*(\mathcal{G}) \neq \mathcal{G}$.

(2) Each $\mathcal{K} \triangleleft \mathcal{G}$ (proper), $F^*(\mathcal{G}) \subseteq \mathcal{K}$ is supersolvable. From (1), $F^*(\mathcal{G}) \neq \mathcal{G}$ we have $\mathcal{K} \triangleleft \mathcal{G}$ (proper), $F^*(\mathcal{G}) \subseteq \mathcal{K}$. As $F^*(\mathcal{G}) \trianglelefteq \mathcal{K} \trianglelefteq \mathcal{G}$, Lemma2.2(c) implies $F^*(F^*(\mathcal{G})) \leq F^*(\mathcal{K}) \leq F^*(\mathcal{G})$. Lemma2.2(d) give $F^*(\mathcal{G}) = F^*(F^*(\mathcal{G})) \leq F^*(\mathcal{K}) \leq F^*(\mathcal{G})$ so $F^*(\mathcal{G}) = F^*(\mathcal{K})$. From Lemma 2.3(a) maximal subgroups of $\mathcal{G}_q \cap F^*(\mathcal{K})$ becomes weakly conjugate $\zeta \cap \mathcal{K}$*−permutable*within $\mathcal{K}$, for all $\mathcal{G}_q \cap \mathcal{K} \in \zeta \cap \mathcal{K}$. So, minimality of $\mathcal{G}$ indicates $\mathcal{K}$ is supersolvable. Consequently (2) hold.

(3) $F^*(\mathcal{G}) = F(\mathcal{G})$.

As $F^*(\mathcal{G}) \neq \mathcal{G}$ from (1), from (2) $F^*(\mathcal{G})$ supersolvable. So, $F^*(\mathcal{G}) = F(\mathcal{G})$ from Lemma 2.2(d).

(4) $\mathcal{G}$ does not has any prime order normal subgroup so, $Z(\mathcal{G})=1$. Suppose $R \trianglelefteq \mathcal{G}$ of prime order. Assume $C_3(R) < \mathcal{G}$ (proper). Using [3, p.36, Theorem 10.6(b)] and (3), $F^*(\mathcal{G}) = F(\mathcal{G}) \leq C_3(R)$. So, $C_3(R)$ becomes supersolvable from (2) moreover, as $\mathcal{G}/C_3(R) \cong \mathrm{Aut}(R)$, thus $\mathcal{G}$ solvable. So, $\mathcal{G}$ becomes supersolvable from Lemma 2.4, a negation. Hence we can consider $C_3(R) = \mathcal{G}$ thus $R \leq Z(\mathcal{G})$. Using (3) and Lemma 2.2(f) $F^*(\mathcal{G}/R) = F^*(\mathcal{G})/R = F(\mathcal{G})/R$. So maximal subgroups of $\mathrm{Syl}(F^*(\mathcal{G}/R)) = F(\mathcal{G})/R$ form weakly conjugate $\zeta R/R$*−permutable w*ithin $\mathcal{G}/R$ (from Lemma 2.6). So, $\mathcal{G}/R$ fulfills assumption of our result also from minimality of $\mathcal{G}$, $\mathcal{G}/R$ becomes supersolvable. In this way $\mathcal{G}$ is supersolvable science order of R is prime, a negation. Consequently, (4) hold.

Suppose Q denote Sylow q-subgroup of $F(\mathcal{G})$, so coming statements are true:

(5) $\Phi(Q)=1$ so, Q is abelian. Assume $\Phi(Q) \neq 1$. Surly, $\Phi(Q)$ is characteristic within Q so $Q \trianglelefteq \mathcal{G}$, $\Phi(Q) \trianglelefteq \mathcal{G}$. From (3) and Lemma2.2(e), $F^*(\mathcal{G}/\Phi(Q)) = F^*(\mathcal{G})/\Phi(Q) = F(\mathcal{G})/\Phi(Q)$. Maximal subgroups of $\mathrm{Syl}(F^*(\mathcal{G}/\Phi(Q)) = F(\mathcal{G})/\Phi(Q))$ weakly conjugate $\zeta\Phi(Q)/\Phi(Q)$*−permutable* within $\mathcal{G}/\Phi(Q)$. So, minimality of $\mathcal{G}$ implies $\mathcal{G}/\Phi(Q)$ becomes supersolvable. As $\Phi(Q) \leq \Phi(\mathcal{G})$, so $\mathcal{G}/\Phi(\mathcal{G})$ is supersolvable. Moreover, class of supersolvable groups becomes F, so $\mathcal{G}$ also supersolvable, negation. Hence, $\Phi(Q)=1$, so Q also abelian.

(6) $QO^q(\mathcal{G}) = \mathcal{G}$. Assume $\mathcal{G} \neq QO^q(\mathcal{G})$. Since $F^*(\mathcal{G}) = F(\mathcal{G}) \leq QO^q(\mathcal{G})$ from (3), $QO^q(\mathcal{G})$ becomes supersolvable from (2) thus $O^q(\mathcal{G})$ supersolvable. So, $\mathcal{G}$ solvable because $\mathcal{G}/O^q(\mathcal{G})$ is a q-group. Lemma

2.4, implies $\mathcal{G}$ becomes supersolvable, negation. So, $QO^{q}(\mathcal{G})=\mathcal{G}$.

(7) $T\cap O^{q}(\mathcal{G})\trianglelefteq\mathcal{G}$ for every $T\leq Q$ (maximum). Suppose $x\in\mathcal{G}$, $S=T^{x^{-1}}$. Surly $T\leq Q$(maximal) as $Q\trianglelefteq\mathcal{G}$ also $|S|=|T|$. From hypothesis, S becomes weakly Conjugate ζ-permutable. So, one can find $\mathcal{N}\vartriangleleft\mathcal{G}$ (subnormal) s.t $S\mathcal{N}=\mathcal{G}$ also $S\cap\mathcal{N}\leq S_1$. Since Q is abelian by (5) moreover $\mathcal{G}=Q\mathcal{N}\Rightarrow Q\cap\mathcal{N}\trianglelefteq\mathcal{G}$. Clearly, $O^{q}(\mathcal{G})\leq\mathcal{N}$ science $\mathcal{N}\vartriangleleft\mathcal{G}$(subnormal) also$|\mathcal{G}:\mathcal{N}|$ becomes power of q. From (6), $\mathcal{N}=QO^{q}(\mathcal{G})\cap\mathcal{N}=(Q\cap\mathcal{N})O^{q}(\mathcal{G})$ thus $\mathcal{N}\trianglelefteq\mathcal{G}$. So, $S_1\mathcal{N}\leq\mathcal{G}$. Trivially, $S\cap S_1\mathcal{N}=S_1(V\cap\mathcal{N})=S_1$, so $S\cap S_1\mathcal{N}$ is ζ-permutable (from Lemma2.7). Thus, $S\cap O^{q}(\mathcal{G})=(S\cap S_1\mathcal{N})\cap O^{q}(\mathcal{G})$ is ζ-permutable from Lemma 2.8 so,$(S\cap O^{q}(\mathcal{G}))\mathcal{G}_q=(T^{x^{-1}}\cap O^{q}(\mathcal{G}))\mathcal{G}_q\leq\mathcal{G}$, $\forall\ \mathcal{G}_q$ *belong to* ζ, $x\in\mathcal{G}$, $\Rightarrow(T\cap O^{q}(\mathcal{G}))\mathcal{G}_q{}^{x}=((T^{x^{-1}})^{x}\cap O^{q}\mathcal{G}^{x})\mathcal{G}_q{}^{x}$ $=(T^{x^{-1}}\cap O^{q}(\mathcal{G})^{x})\,\mathcal{G}_q{}^{x}=(T^{x^{-1}}\cap O^{q}(\mathcal{G})\mathcal{G}_q)^{x}\leq\mathcal{G}$, $\forall\ x\in\mathcal{G}$, $\mathcal{G}_q\in\zeta$. Science Sylow subgroups are conjugate, so $T\cap O^{q}(\mathcal{G})$ becomes S-permutable within $\mathcal{G}$. From [2, p.17, Lemma 1.2.16] $O^{q}(\mathcal{G})\leq N_{\mathcal{G}}(T\cap O^{q}(\mathcal{G})$. Moreover, $T\cap O^{q}(\mathcal{G})\trianglelefteq QQ$ is abelian from (5). So, $\mathcal{G}=QO^{q}(\mathcal{G})\leq N_{\mathcal{G}}(T\cap O^{q}(\mathcal{G}))$ from (6) thus $T\cap O^{q}(\mathcal{G})\trianglelefteq\mathcal{G}$. Hence (7) is satisfied.

(8) $Q\cap O^{q}(\mathcal{G})\neq 1$. Assume $Q\cap O^{q}(\mathcal{G})=1$. Since $[Q\ ,\ O^{q}(\mathcal{G})]\leq Q\cap O^{q}(\mathcal{G})=1$, it follows that $O^{q}(\mathcal{G})\leq C_3(Q)$. Moreover, $Q\leq C_3(Q)$ from (5) Q is abelian. So,$\mathcal{G}=QO^{q}(\mathcal{G})\leq C_3(Q)$ from (6) thus $Q\leq Z(\mathcal{G})$, a negation to (4). Hence, (8) is clear.

(9) $Q\cap\Phi(\mathcal{G})\neq 1$. Consider $Q\cap\Phi(\mathcal{G})=1$. By Lemma 2.9, $Q=Y_1xY_2x...Y_n$, where each $Y_k\trianglelefteq\mathcal{G}$ (minimal), for $k=1, 2... n$. Since $Q\cap O^{q}(\mathcal{G})\neq 1$ by (8) also $\Phi(Q)=1$ from (5), so $T\leq Q$(maximal) s.t $Q=(Q\cap O^{q}(\mathcal{G}))T\Rightarrow|(Q\cap O^{q}(\mathcal{G})):(T\cap O^{q}(\mathcal{G}))|=|Q:T|=q$. So, $T\cap O^{q}(\mathcal{G})\trianglelefteq\mathcal{G}$ by (7). Thus, $(Q\cap O^{q}(\mathcal{G}))/(T\cap O^{q}(\mathcal{G}))$ becomes chief factor of $\mathcal{G}$ having order q. Suppose an operator group Q having operator domain $\Omega=Inn(\mathcal{G})$. So, $1\trianglelefteq Y_1\trianglelefteq Y_1Y_2\trianglelefteq...\trianglelefteq Y_1Y_2...Y_n=Q$ be $\Omega-$*composition* series of $\Omega-$*group* Q. As $(Q\cap O^{q}(\mathcal{G}))/(T\cap O^{q}(\mathcal{G}))$ be Ω-composition factor of Q, so, $(Q\cap O^{q}(\mathcal{G}))/(T\cap O^{q}(\mathcal{G}))\cong Y_n$ *(* by Jordan-Holder Theorem*)*, for few k $(1\leq k\leq n)$. Therefore, $|Y_k|=q$, for few $k(1\leq k\leq n)$, a negation to (4). Hence *(9)* hold. Suppose $Y\leq\mathcal{G}$ (minimal), $Y\subseteq Q\cap\Phi(\mathcal{G})$, so coming are true:

(10) $F(\mathcal{G}/Y)=F(\mathcal{G})/Y$. Assume $\mathcal{K}/Y=F(\mathcal{G}/Y)$. So $\mathcal{K}\trianglelefteq\mathcal{G}$ (nilpotent) from [6, p.270, Satz 3.5] thus $\mathcal{K}\leq F(\mathcal{G})$. Hence, (10) is true.

(11) $Y\trianglelefteq\mathcal{G}$(unique minimal), $Y\subseteq Q$. Lemma 2.2(a), (10) implies $F^{*}(\mathcal{G}/\mathcal{N})=F(\mathcal{G}/\mathcal{N})E(\mathcal{G}/\mathcal{N})=F(\mathcal{G})/Y\,W/Y$ and $[F(\mathcal{G})/Y,\ W/Y]=1$, $W/Y=E(\mathcal{G}/Y)$ denote layer subgroup of $\mathcal{G}/Y$. As $[F(\mathcal{G})/Y,\ W/Y]=1$, so $[F(\mathcal{G}),\ W]=Y$. Suppose $\mathcal{K}\trianglelefteq\mathcal{G}$ (minimal), $\mathcal{K}\subseteq Q$ different from Y. Thus $[\mathcal{K},\ ]\leq\mathcal{K}\cap Y=1$ so $W\leq C_G(\mathcal{K})$. From (4), $C_G(\mathcal{K})<\mathcal{G}$(proper). As $F^{*}(\mathcal{G})=F(\mathcal{G})\leq C_{\mathcal{G}}(\mathcal{K})$ thus (3), [3, p.36, Theorem 10.6(b)] implies $C_{\mathcal{G}}(\mathcal{K})$becomes supersolvable from (2) so W also solvable. As W/Y solvable perfect group from Lemma 2.2(b), so $W/Y=1$ implies $F^{*}(\mathcal{G}/Y)=F(\mathcal{G})/Y$. From Lemma 2.6, maximal subgroups of $Syl(F^{*}(\mathcal{G}/Y)=F(\mathcal{G})/Y)$ becomes weakly ζ $Y/Y-$*permutable* within $\mathcal{G}/Y$. So, $\mathcal{G}/Y$ fulfill assumption of theorem thus $\mathcal{G}/Y$ becomes supersolvable from minimality of $\mathcal{G}$. As $Y\leq\Phi(\mathcal{G})$ implies$\mathcal{G}/\Phi(\mathcal{G})$ becomes supersolvable. It indicates $\mathcal{G}$ also supersolvable science class of supersolvable groups form saturated formation, a negation. Hence (11) is satisfied.

(12) Final negation (5) implies, $\mathscr{M}$≤Q(maximal) s.t Q=Y$\mathscr{M}$. When Y≤$\mathscr{M}$∩$O^q(\mathcal{G})$, then Y≤$\mathscr{M}$ so Q=$\mathscr{M}$,a negation. Therefore, Y⊈$\mathscr{M}$∩$O^q(\mathcal{G})$. As $\mathscr{M}$∩$O^q(\mathcal{G})$⊴ $\mathcal{G}$ by (7) also Y≠ $\mathscr{M}$∩$O^q(\mathcal{G})$, (11) implies $\mathscr{M}$∩$O^q(\mathcal{G})$=1. Surly, Q=(Q∩$O^q(\mathcal{G})$)$\mathscr{M}$ science Y≤Q∩$O^q(\mathcal{G})$ from(8),(11). Hence, |Q∩$O^q(\mathcal{G})$|=|Q:$\mathscr{M}$|=q, a negation to (4). It finishes the proof.

Proof of Theorem 1.4.
1⇒2 When $\mathcal{G}$∈ Ƒ, so (b) is correct along Ⴙ= 1.

2⇒1 Maximal subgroups of $\mathcal{G}_q$∩$Ƒ^*(\mathcal{K})$ are weakly conjugate *Z∩𝒦−permutable* within $\mathcal{K}$,∀ $\mathcal{G}_q$∩$\mathcal{K}$ ∈ Z∩$\mathcal{K}$ (from Lemma2.3(a)). Theorem1.3 indicate $\mathcal{K}$ is supersolvable so $Ƒ^*(\mathcal{K})$=Ƒ($\mathcal{K}$). So, $\mathcal{K}$⊴$\mathcal{G}$ (solvable) along $\mathcal{G}/\mathcal{K}$∈Ƒ and maximal subgroups of Syl(Ƒ($\mathcal{K}$)) becomes weakly Conjugate ζ-permutable within $\mathcal{G}$. Lemma 2.4 implies $\mathcal{G}$ ∈ Ƒ. It finishes the proof.

The coming famous results available in literature are at once outcomes of Theorems 1.3, 1.4.

Corollary 3.1 [8, Theorem 3.1] Suppose $\mathcal{K}$⊴$\mathcal{G}$ s.t $\mathcal{G}/\mathcal{K}$∈ μ. When maximal subgroups of $\mathcal{G}_q$∩$Ƒ^*(\mathcal{K})$ becomes ζ-permutable within $\mathcal{G}$, for every $\mathcal{G}_q$ ∈ ζ, so $\mathcal{G}$∈ μ .

Corollary 3.2 [9, Theorem 3.1] Suppose $\mathcal{K}$⊴$\mathcal{G}$ s.t $\mathcal{G}/\mathcal{K}$∈ μ. When maximal subgroups of Syl($Ƒ^*(\mathcal{K})$) are S-permutable within $\mathcal{G}$, next $\mathcal{G}$ ∈ μ.

Corollary 3.3 [8, Main Theorem] Suppose *saturated formation* Ƒ *carrying every types of supersolvable groups* μ. So, $\mathcal{G}$∈ Ƒ iff $\mathcal{K}$⊴$\mathcal{G}$ s.t $\mathcal{G}/\mathcal{K}$∈Ƒ, maximal subgroups of $\mathcal{G}_q$∩$Ƒ^*(\mathcal{K})$ becomes ζ-permutable within $\mathcal{G}$, for each $\mathcal{G}_q$ ∈ ζ.

Corollary 3.4 [9, Theorem 3.4] For Ƒ, μ *and* $\mathcal{G}$. When $\mathcal{G}$⊴$\mathcal{K}$ s.t $\mathcal{G}/\mathcal{K}$∈ Ƒ and maximal subgroups of the Syl($Ƒ^*(\mathcal{K})$) are S-permutable within $\mathcal{G}$, next $\mathcal{G}$∈ Ƒ.

Corollary 3.5 [10, Theorem 3.1] For Ƒ, μ *and* $\mathcal{G}$. When $\mathcal{K}$⊴$\mathcal{G}$ s.t $\mathcal{G}/\mathcal{K}$∈ Ƒ, maximal subgroups of Syl($Ƒ^*(\mathcal{K})$) becomes c-normal within $\mathcal{G}$, next $\mathcal{G}$∈ Ƒ.

Data Availability
In this article, no data were utilized.

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