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On Nearly Hall S-Semi Embedded Subgroups and *P*-Nilpotency of Finite Groups

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Abstract
A subgroup $S_1 \leq G$, is said to be nearly Hall s-semiembedded within G (N_{HSseG}) if there exits $S_2 \trianglelefteq G$ such that S_1S_2 is S-semiembedded in $G(s_{seG})$ and $S_1 \cap S_2 \leq (S_1)_{nSG}$, where $(S_1)_{nSG}$ is created by those subgroups of S_1 which are Hall S-semiembedded in G (H_{SseG}). In this paper, we investigate new notion of nearly Hall s-semiembedded subgroups on the structure of a finite group. Moreover, some new criteria were established for p-nilpotency and supersolvability of a finite group using nearly Hall s-semiembedded subgroups. Our results about formations are extension of some earlier theorems available in literature.

Keywords: S-Quasinormal Subgroups, Hall S-Semiembedded Subgroups, S-Semiembedded Subgroups, *P*-Nilpotent

Introduction

In this paper, we study a class of only finite groups. All groups under consideration are saturated as it can be seen in [3] and [5]. The most of the notations and terminologies are canonical. The Sylow subgroup of G is denoted by $syl(G)$ and maximal subgroup of M is denoted by max(M). Here, $\pi(G)$ denotes set of primes dividing $|G|$and $|G|_p$ denotes the order of Sylow q-subgroups of G. A subgroup $\mathcal{H} \leq G$ is known as S-permutable in $G(s_{perG})$ [10] if $\mathcal{H}$permutes with each $syl(G)$. Another generalization is S-permutability, which is given by Petrillo [8] as: a subgroup $\mathcal{H} \leq G$ is said to be s-semipermutable in G[1] if $\mathcal{H}G_p = G_p\mathcal{H}$ for any G with $(p, |\mathcal{H}|) = 1$. Recently, Guo and Li [7] introduced an extension of s-semiembedded subgroups. A subgroups $\mathcal{H}$ of G is called S-semiembedded (S_{seG}) if there exist S_{perG}, $\mathcal{K}$ in G such that $\mathcal{HK}$ is S_{perG} and $\mathcal{H} \cap \mathcal{K} \leq \mathcal{H}_{\mathcal{H}G}$ and S_{sperG} contained in $\mathcal{H}$. A subgroup $\mathcal{H} \leq G$ is said to be S-embedded within G[6] if $\mathcal{R} \trianglelefteq G$ such that $\mathcal{HR}$ is s_{perG} and $\mathcal{H} \cap \mathcal{R} \leq \mathcal{H}_{sG}$, where $< \mathcal{H}_{sG} > \leq \mathcal{H}$ that are s_{perG}. Note that $\mathcal{H}_{sG}$ is s_{perG} from [14, Lemma 2.8]. Another idea investigated is c-normal given by Guo et al., [5]. Recently, Mao et al., [6] introduced the idea of s-semiembedded subgroups of finite groups. In [7], the concept of Hall s-semiembedded subgroup was introduced by Guo and Li. A subgroup $\mathcal{K}$ is a Hall S-semiembedded (H_{SseG}) if there exists Hall subgroup for $P \in Syl_p(G)$, where $(p, |\mathcal{K}|) = 1$. The researchers obtained many different and exceptional conclusions by using the above ideas, (see, [6, 8, 16, 17, 18 and 19]).
In this paper we integrated all above describe concepts and present the following our new notion as follows.

Definition1.1. A subgroup $S_1 \leq G$, is said to be nearly Hall s-semiembedded within G (N_{HSseG}) if there exits $S_2 \trianglelefteq G$ such that S_1S_2 is S-semiembedded in $G(s_{seG})$ and $S_1 \cap S_2 \leq$

$(S_1)_{nSG}$, where $(S_1)_{nSG}$ is created by those subgroups of S_1 which are Hall S-semiembedded in G (H_{SseG}).

Example1.2. Consider $G = S_5$ be a group. Since $K_1 =< 1 \ \ 2 >$ is known as nearly Hall s-semiembedded within G (N_{HSseG}) if $K_2 = A_5 \trianglelefteq G$ such that $K_1 \cap K_2 \leq (K_1)_{nSG}$, $(K_1)_{nSG}$ is the subgroups of K_1which are $H_{SseG.}$
Now we classify some nearly Hall s-semiembedded subgroups on representation of a finite group G.

2 Preliminaries

In this section, we give some lemmas which indeed, very effective to prove our coming theorems in next section.

Lemma 2.1.[4, 9] Suppose K_1 be S-permutable in G (s_{perG}) and $K_2 \trianglelefteq G$, then:
(1) If $K_3 \leq G$, $K_1 \cap K_3$ is s_{perK_3}.
(2) K_1K_2 and $K_1 \cap K_2$ are s_{perG}, and K_1K_2/K_2 is s_{perG/K_3}.
(3) $K_1 \subseteq G$, where K_1 subnormal subgroup.
(4) If K_1 is a q-group, then $N_G \geq O^q(G)$, for few prime q.

Lemma 2.2. [4, 9] Suppose $\mathcal{K} \leq L \leq G$.
(1) If $\mathcal{K} \leq G$ is s-semipermutable, then $\mathcal{K} \leq L$ becomes s-semipermutable.
(2) Consider $I \trianglelefteq G$, and $\mathcal{K}$ be q-group. If $K \leq G$ is $s_{per\ G}$ then KI/I is permutable within G/I.

Lemma 2.3. [7] Let $I \leq G$ is Hall ss-Embedded subgroup within G. Then
(1) If $I \leq J \leq G$, then $I \leq J$ is Hall ss- Embedded.
(2) If I is a q-group, $K \trianglelefteq G$ consequently $IK/K \leq G/K$ is H_{SseG}.

Lemma2.4. [4] Assume J be subnormal q-subgroup. If $J \leq G$ is s-semipermutable, subsequently $J \leq G$ is s_{perG}.

Lemma2.5. Consider $S_1 \leq S_2 \leq G$ and $S_3 \trianglelefteq G$.
(1) If S_1 is N_{HSseG}, then $Q \leq S_2$ is nearly Hall ss-Embedded.
(2) Consider S_1 is nearly Hall ss-Embedded q-subgroup and $S_3 \leq S_1$ or $(q, |S_3|) = 1$, then $S_1 S_3/S_3$ is nearly Hall ss-Embedded within G/S_3.
(3) $S_2 \trianglelefteq G$, and S_1 is N_{HSseG}, then $\exists$ some $S_3 \trianglelefteq G$ in S_2 such that $S_1S_3 \leq G$ is S-permutable and $S_1 \cap S_3 \leq (S_1)_{SG}$.
(4) If S_1 is $N_{HSseG}$and $S_1 \leq F(G)$, then $S_1 \leq G$ is S-embedded.

Proof. Assume some normal subgroup $Q \leq G$, with $S_1Q \leq G$ is s_{perG} and $S_1 \cap Q \leq (S_1)_{SG}$. So:
(1) $S_2 \cap Q \trianglelefteq S_2$, Lemma 2.1 and Lemma 2.2 shows $S_1(S_2 \cap Q) = S_2 \cap S_1Q$ is s_{perS_3} and $S_1 \cap (S_2 \cap Q) = S_1 \cap Q \leq (S_1)_{SG} \leq (S_1)_{SS_1}$.Thus, S_1 is nearly Hall ss-Embedded within S_1.
(2) Since $QS_3/S_3 \trianglelefteq G/S_3$ and $(S_1S_3/S_3)(QS_3/S_3) = S_1QS_3/S_3$ is $s_{per\ G/S_3}$. If $S_3 \leq S_1$, then $S_1/S_3 \cap QS_3/S_3 = (S_1 \cap Q)S_3/S_3 \leq (S_1)_{SG}\ s_3/s_3$. If S_3 is a q'-group, so $|S_1 QS_3|_q = \frac{|S_1|.|QS_3|_q}{|SHQ|_q} = \frac{|S_1|.|Q|q}{|S_1Q|q} = |S_1 \cap Q| \Rightarrow S_1 \cap QS_3 = S_1 \cap Q$. We see
$(S_1S_3/S_3) \cap (QS_3/S_3) = (S_1S_3 \cap QS_3)/S_3 = (W_1 \cap HS_3)S_3/S_3 = (S_1 \cap Q)S_3/S_3 \leq (S_1)_{SG}\ s_3/s_3$. From Lemma 2.3, it is clear that $(S_1)_{nSG}\ s_3/s_3$ is Halls-semipermutable in G/s_3. So, S_1S_3/S_3 is N_{HSseG/S_3}.
(3) Assume $S_3 = S_2 \cap Q$, so $S_1S_3 = S_1(S_2 \cap Q) = S_2 \cap S_1Q$ lies within G *is* S-permutable and $S_1 \cap S_3 = S_1 \cap (W_2 \cap Q) = S_1 \cap Q \leq (S_1)_{SG}$.

(4) From Lemma 2.4 and definition condition (4) is satisfied clearly.

Lemma 2.6. [10] Let a prime q divided $|G|$ satisfying $(|G|, q-1) = 1$. Then
(1) If $R \trianglelefteq G$ and $|R| = q$ order, then $R \subseteq Z(G)$.
(2) G *is* q-nilpotent, if G has cyclic $Syl_q(G)$.

Lemma2.7. [11] Assume I, J and K contained in G. Then
(1) $I \cap JK = (I \cap J)(I \cap K)$.
(2) $IJ \cap IK = I(J \cap K)$ are equivalent.

3 Proofs of theorems

In this section, we prove our main results.

Theorem 3.1. Assume $|G|$ satisfying $(|G|, \wp - 1) = 1$, and $\wp$ be prime divisor and $\mathcal{K}$ *be* a Syl_ρ(G). Suppose each max($\boldsymbol{U}$) not a ρ-nilpotent is nearly Hall s-semiembedded in G, then G becomes ρ-nilpotent.

Proof. We take an example to prove the result. The example will be of minimal order which prove result is not true.
(1) $\mathcal{K}$ is not cyclic:
As $(|G|, \wp - 1) = 1$ by Lemma 2.6 (2), assume $\mathcal{K}$ is not cyclic. Consider a proper subgroup $W_1 \leq \mathcal{K}$ has ρ-nilpotent supplement. If G is not ρ-nilpotent, assume W_2 is not ρ-nilpotent containing $\mathcal{K}$ and each suitable proper subgroup in W_2 is ρ-nilpotent. So by [8, IV Theorem 5.4], and W_2 is minimal not nilpotent such that:
(i) $W_2 = [\mathcal{K}](W_2)_{\wp}$ and for normal $\mathcal{K} = Syl_\rho(G)$ and $(W_2)_{\wp}$ a non-normal cyclic $Syl_{\wp}(W_2)$.
(ii) $\mathcal{K}/\Phi(\mathcal{K})$ is minimal of $W_2/\Phi(\mathcal{K})$.

(2) Each proper subgroup $\mathcal{K}$ has no ρ-nilpotent:
Since $G = W_1W, W_2 = W_2 \cap W_1W = W_1(W_2 \cap W)$.The facts $W_2 \cap W \leq W$ is $\wp$-nilpotent but W_2 is not $\wp$-nilpotent$\Rightarrow \mathcal{J} = W_2 \cap W \leq W_2$ for $\mathcal{J} \leq W_2$.So $\mathcal{J}$ −nilpotent and $\mathcal{J} = Syl_{\wp}(\mathcal{J}) \times Syl_{\wp'}(\mathcal{J})$. Since $\mathcal{K} = W_1Sly_\rho(\mathcal{J})$, $Sly_\rho(\mathcal{J})\Phi = \Phi(D)$. AssumeT$_2/\Phi$, assume the reality of $Syl_{\rho'}(\mathcal{J}) \leq (N_1)_{T_2}(Sly_\rho(\mathcal{J})) \Rightarrow Sy\wp_{\rho'}(\mathcal{J})\Phi/\Phi \leq (W_1)W_2/\Phi(Sly_{\wp}(\mathcal{J})\Phi/\Phi)$. Also, $Sly_{\wp}(\mathcal{J})\Phi/\Phi \trianglelefteq \mathcal{K}/\Phi$. As $\mathcal{K}/\Phi$ is an abelian group. So, $Sly_{\wp}(\mathcal{J})\Phi/\Phi \trianglelefteq Syl_{\wp'}(\mathcal{J})\Phi/\Phi, \mathcal{K}/\Phi) = W_2/\Phi$. Since $Sly_{\wp}(\mathcal{J})\Phi/\Phi \neq 1^{-} and\ \mathcal{K}/\Phi$ is chief factor of $W_2, Sly_{\wp}(\mathcal{J})\Phi/\Phi = \mathcal{K}/\Phi$. It follows $Sly_{\wp}(\mathcal{J}) = \mathcal{K}$. So, $\mathcal{J} = W_2$. As required result (1).

(3) G does not contain non-abelian and simple group:
Suppose $\mathcal{K}_1$ be max($\mathcal{K}$), by hypothesis and by (1) we have $\mathcal{K}_1$ is nearly Hall S-semiembedded within G.Thus $W \trianglelefteq G$ s.t. $\mathcal{K}_1W$ is s_{perG} and $\mathcal{K}_1 \cap W \leq (\mathcal{K}_1)_{nsG}$.When G simple and non-abelian, then $W = G$ or 1. If $W = 1$ thus $D_1 = D_1T$ is S-permutable. So $D_1 \leq G$, so negation to assumption. Consequently $T = G$ and$(D_1)s_G$, thus assume any max($\mathcal{K}$) is S-semipermutable. Suppose $W_2 \leq \mathcal{K}$ non-trivial, assume $N_G(W_2)$. Moreover, $S_1 \in Syl_{\wp}(N_G(W_2))$ and $U_1 \in Syl_{\wp}(N_G(W_2))$ for each $\wp' \neq \wp$. Assume U be $Syl_{\wp'(G)}$ containing U_1, then each max($\mathcal{K}$)& U are permutable. Since $\mathcal{K}$ is not cyclic, $\mathcal{K} = \mathcal{K}_1\mathcal{K}_2$ hold for some max($\mathcal{K}_1$)and $\mathcal{K}_2$ of $\mathcal{K}$. Thus $\mathcal{K}U = \mathcal{K}_1\mathcal{K}_2U = U\mathcal{K}_1\mathcal{K}_2 = U\mathcal{K}$ Hall subgroup (proper). Surly $\mathcal{K}U$ satisfied. So minimality $\Rightarrow$ $\mathcal{K}U$ is $\wp$-nilpoten. Hence $U \trianglelefteq \mathcal{K}U$ and $U_1 = U \cap N\mathcal{K}U(W_2) \trianglelefteq N\mathcal{K}U(W_2)$. Obviously, $(\mathcal{K}_2)U_1 = W_2 \times U_1$ satisfied for each Sylow U_1 of $N_G(W_2)$ for $\wp$. Hence $N_G(W_2)$ becomes $\wp$-nilpotent. Thus from [8, IV, Theorem 5.8] G cannot be as required in (3).

(4) $\Phi(G) = 1$ and $N_1 \trianglelefteq N_1G, G/N_1$ is $\wp$-nilpotent:

Surly, $WN_1/N_1 \trianglelefteq G/N$ and $\mathcal{K}_1N_1/N_1.WN_1/N_1 = \mathcal{K}_1WN_1/N_1$ is S-permutable in G/N_1. Further, we obtain a $Syl_{p}(N_1)$ as $\mathcal{K}_1 \cap N_1$. Thus $|(\mathcal{K}_1 \cap N_1)(W \cap N_1)|_{p} = |\mathcal{K}_1 \cap N_1| = |N_1|_{p}\, |N_1\ \mathcal{K}_1W|_{p}$. Since $\mathcal{K}_1$ is a p-group so,
$|N_1 \cap \mathcal{K}_1W|_{p'}.(|N_1|_{p'}. |\mathcal{K}_1W|_{p'})/(|N_1\ \mathcal{K}_1 W|_{p'})$
$= \frac{|N_1|_{p'}.|W|_{p'}}{|NW_1|_{p'}} = |N_1 \cap W|_{p'}.(\mathcal{K}_1 \cap N_1)(W \cap N_1)|_{p'}$.
Above equation indicates $(N_1 \cap \mathcal{K}_1)(N_1 \cap W) = N_1 \cap \mathcal{K}_1W$. From Lemma 2.7, $\mathcal{K}_1N_1 \cap WN = (\mathcal{K}_1 \cap W)N_1$. consequently, $\mathcal{K}_1N_1/N_1 \cap WN_1/N_1 = (\mathcal{K}_1 \cap W)N_1/N_1 \leq (\mathcal{K}_1)_{nSG}\, N_1/N_1$, from Lemma2.2 $(\mathcal{K}_1)_{nSG}\, N_1/N_1$ is S-semipermutable within G/N_1. Therefore, G/N_1 is satisfied and by minimality of G it is p-nilpotent. As p-nilpotent group class is saturated class, $N_1 \leq G$, N_1 is minimal and unique subgroup, and $\Phi(G) = 1$.
(5) $O_{p'}(G) = 1$:
If $O_{p'}(G)$ not equal to1 next $N_1 \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is p-nilpotent from (4). So G is p-nilpotent, which is against our supposition.
(6) $O_{p}(G) = 1$ & N_1 be p-nilpotent:
If $O_{p}(G) \neq 1$, so $N_1 \leq O_{p}(G)$. As $\Phi(G)$ is 1, and G has a max(M) s.t. $G = [N_1]M$. Since $O_{p}(G) \leq F(G) \leq C_G(N_1)$ and $C_G(N_1) \cap M \trianglelefteq G, N_1$ from uniqueness of N_1 indicate $N_1 = O_{p}(G)$. Since $\mathcal{K} = N_1(\mathcal{K} \cap M)$ *and* $N_1 \cap M = 1$, $\mathcal{K} \cap M$ is $Syl_{p}(M)$ and $\exists$, a max$(\mathcal{K}_1) \leq \mathcal{K}$ such that $\mathcal{K} \cap M \leq \mathcal{K}_1$ and $\mathcal{K} = N_1\mathcal{K}_1$. By (3), as $\mathcal{K}_1$ is H_{SseG}. So $W \trianglelefteq G$ such that $\mathcal{K}_1W$ is $s_{per\,G}$, also $\mathcal{K}_1 \cap W \leq (\mathcal{K}_1)_G$. If W is one, and $\mathcal{K}_1 = \mathcal{K}_1W$ is $s_{per\,G}$. Clearly $\mathcal{K}_1$ is subnormal in G and then $\mathcal{K}_1 \leq O_{p}(G) = N_1$. Then we deduce that, $\mathcal{K} = \mathcal{K}_1N_1 = N_1$(minimal normal). Since $N_G(\mathcal{K}_1) \geq O^{p}(G)$ from Lemma 2.1(4) and $\mathcal{K}_1 \trianglelefteq \mathcal{K}, \mathcal{K}_1 \triangleleft G$(proper) lies in $\mathcal{K} = N_1$, a negation. Consequently $W \neq 1$ and $N_1 \leq W$. So $N \cap \mathcal{K}_1 = N_1 \cap \mathcal{K}_1 \cap W = N_1 \cap (\mathcal{K}_1 \cap W)U \trianglelefteq (\mathcal{K}_1 \cap W)U$ for each $Syl_{p'}(U)$ of G for $p' \neq p$. Consequently $U \leq (N_1)_G(N_1 \cap \mathcal{K}_1)$ clear for $Syl_{p'}(U)$ of G with $p' \neq p$. As $N_1 \cap \mathcal{K}_1 \trianglelefteq \mathcal{K}$ also normal in G. Hence $N_1 \cap D_1 = 1$ and $|N_1| = D$ from Lemma 2.6 (1), N_1 lies $Z(G)$. As G/N_1 is nilpotent as well G also p-nilpotent, a negation. $(N_1)_{p}\, charN_1 \trianglelefteq G$ as N_1 is ρ-nilpotent so, $Syl_{p}(N_1) \leq O_{p}(G) = 1$ by (4). Which is wrong since N_1 is p-group and $N_1 \leq O_{p}(G) = 1$.
(7) Final result:
By Lemma 2.5, since $\mathcal{K}N_1$ satisfied hypothesis. So, $\mathcal{K}N_1$ is p-nilpotent if $\mathcal{K}N_1 < G$. Then obvious N_1 is p-nilpotent, contradict to (6). Thus $G = \mathcal{K}N_1$. As N_1 is not soluble, and $N_1 = S_1 \times S_2 \times \ldots \times S_k$ is simple and non-abelian S_j. From (2), as it is clear $N_1 < G$ and $\mathcal{K} \cap N_1 < K$. Let $S_{p} \in Syl_{p}(S_1) \leq \mathcal{K}_1$ for some max$(\mathcal{K}_1)$ of $\mathcal{K}$. By assumption and (1), as we have, $W \trianglelefteq G$ s.t. $\mathcal{K}_1W$ is s_{perG} and $\mathcal{K}_1 \cap W \leq (\mathcal{K}_1)_G$. If $T = 1$, then $\mathcal{K}_1$ is s_{perG} and $O_{p}(G) \neq 1$, which negates to (6). Thus $W \neq 1$ so $N_1 \leq W$. If $\mathcal{K}_1 \cap W = 1$, next $|W| \leq p$. Thus W be p-nilpotent see Lemma2.6(2), N_1 is also p-nilpotent. Obviously $\mathcal{K}_1 \cap W \neq 1$. Assume p' is divisor of $|G|$ distinct from p and U a $Syl_{p'}(G)$. Then:
$|U \cap \mathcal{K}_1W| = |U|.|\mathcal{K}_1W|_{p'}/|U\mathcal{K}_1W|_{p'} = |U|.|W|_{p'}/|UW|_{p'} = |U \cap W| = |(U \cap \mathcal{K}_1)(U \cap W)|$.
The above expression indicates $U \cap \mathcal{K}_1W = (U \cap \mathcal{K}_1)(U \cap W)$. Then $U\mathcal{K}_1 \cap UW = U(\mathcal{K}_1 \cap W)$ from Lemma2.7 so, $N_1 \cap \mathcal{K}_1U = N_1 \cap (\mathcal{K}_1U \cap WU) = N_1 \cap (\mathcal{K}_1 \cap W)U$. Thus $S_1 \cap (\mathcal{K}_1 \cap W) = S_1 \cap \mathcal{K}_1 = S_{\rho}$ *is* $Syl_{p}(S_1)$. We have a non-abelian N_1, for $p = 2$. So $p' \neq 2$ is divisor of $|S|$, non-abelian and simple S_1 contains Hall $\{2, p'\}$, negation to [12, Lemma2.6]. Each supersolvable group G for its smallest divisor p is p-nilpotent, G having any subgroup W_1 is not a p-nilpotent supplemented within G moreover, G has no supersolvable supplement. Thus, from Theorem 3.1, we concluded the followings:

Corollary3.2. Let $\mathcal{T}$ be $Syl_{p'}(G)$, where $p' = min\ \pi(G)$. If each max$(\mathcal{H})$ without supersolvable supplement within G is nealy Hall s-semiembedded within G (N_{HSseG}), then G is nilpotent.

Theorem 3.3. Assume $\mathcal{F}$ is saturated formation and G consisting of solvable groups μ. Then $G \in \mathcal{F}$ if and only if $G \trianglelefteq \mathcal{R}$ such that $G/\mathcal{R} \in \mathfrak{F}$ for every non-cyclic $Syl_Q(\mathcal{R})$, every *max(Q)* without supersolvable supplemented within G is nearly Hall s-semiembedded within G.

Proof. Compulsory part of our theorem is clear, but it remains to prove second part. Consider it is false statement and we take counter example $(G,\ \mathcal{R})$, where $|G|\ |\mathcal{R}|$ minimal.

(1) $\mathcal{R}$ is $\wp'$-nilpotent for its lowest divisor $\wp'$:

Le $\wp'=min\ \pi(\mathcal{R})$, be $Syl_{\wp'}(\mathcal{R})$ $\mathcal{R}$ is $\wp'$-nilpotent when Q is cyclic from [Theorem2.8, IV, 8]. Assume Q is not cyclic by Lemma2.4, as each $\max(Q)$ without supersoluble supplement lies within $\mathcal{R}$ is $N_{HSse\mathcal{R}}$.

So Corollary3.2$\Rightarrow$ $\mathcal{R}$ must be $\wp'$-nilpotent.

(2) $\mathcal{R}\ = Q$ is not cyclic:

Assume $P < \mathcal{R}$, N_1be normal $\wp'$-complement of $\mathcal{R}$. So $N_1 \trianglelefteq G$ and by Lemma2.5(2), our assumption is satisfied for $(G/N_1)(\mathcal{R}/N_1)$. So $G/N_1 \in \mathfrak{F}$ from minimal of G. We conclude assumption is true for (G, N_1). Thus $(G, \mathcal{R}\ )$ reveals $N_1 = 1$ and so $\mathcal{V} = \mathcal{R}$. As $G/\mathcal{R} \in \mathfrak{F}$, by [16, Lemma2.16] suppose $\mathcal{R}$ is not cyclic.

(3) From minimal normal Q subgroup, $G^F = Q$:

Assume N be $\min(G)$ in Q, from Lemma 2.5, assumption fulfill for G/H. So $G/H \in \mathfrak{F}$ by minimality G. Next, H is only $\min(G)$ such that $G = [H]M$. Then $Q = Q \cap HM = H(Q \cap M)$. As $Q \leq \mathfrak{F}(G) \leq C_G(H), (Q \cap M) \trianglelefteq G$ and $Q \cap M = 1$. So, $Q =\ N =\ G^F$ is normal $\min(G)$.

(4) Final negation:

Assume P_1 be $\max(P) = \mathcal{W}$. If P_1 has supersoluble supplement $K_1 \leq G$, then $PK_1 = G$ and $1 \neq P \cap K_1 \trianglelefteq G$. So $P \cap K_1 = P, G$ is K_1 supersoluble, against our supposition. Thus, P_1 is N_{HSseG}. Consider $T \trianglelefteq G$ lies in P such that P_1T is S-permutable also $P_1 \cap T \leq (P_1)_G$. As P is $\min(G), T = 1$ or P. When $T = 1$, $P_1 = P_1T$ is S-permutable. If $T = P$, then $P_1 = P_1 \cap T = (P_1)_G \cap T$ is Hall s-semipermutable by Lemma 2.1(4), it also be observe that P_1 is S-permutable from Lemma 2.1(4), so $O^{\wp}(G) \leq N_G(P_1)$. So for any $\max(P_1)$ of P, $|G : N_G(P_1)| = \wp^a$, a is any integer. Let $M_1, M_2 \ldots M_t$ having $\max(P)$, next $\wp | t$ negation to [8, III, 8.5(d)]. Hence proved.

Conclusion

In this paper, we have studied some characterization of nearly Hall s-semiembedded subgroups of a finite group G. We have prove that for prime divisor $\wp\ for\ |G|$ satisfying $(|G|, \wp\ -\ 1) = 1$, and $\mathcal{K}\ be$ a Syl_ρ(G). Suppose each $\max(U)$ not a ρ-nilpotent is nearly Hall s-semiembedded in G, then G is ρ-nilpotent. Further, we give a necessary and sufficient condition for a group G to be formation if a non-cyclic Sylow-p subgroups with each maximal by considering nearly Hall S-semiembedded subgroups.

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